

Alternative Route to Strong Interaction: *Narrow* Feshbach Resonance

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We show that a narrow resonance produces strong interaction effects far beyond its width on the side of the resonance where the bound state has not been formed. This is due to a resonance structure of its phase shift, which shifts the phase of a large number of scattering states by π before the bound state emerges. As a result, the magnitude of the interaction energy when approaching the resonance on the “upper” and “lower” branch from different side of the resonance is highly asymmetric, unlike their counter part in wide resonances. Measurements of these effects are experimentally feasible.

Strongly interacting Fermi gases are known for their universal properties and their “high” superfluid transition temperature (T_c), in the sense that the ratio T_c/T_F is highest among all known superfluids, where T_F is the Fermi temperature. Since strong interactions are often the source of macroscopic quantum effects, it is useful to have more examples of strongly interacting systems. By “strongly interacting”, we mean systems whose interaction energy is a significant part of the total energy at low temperatures. The interacting Fermi gases studied today are mostly associated with the so-called “wide” resonances, whose widths are much larger than the Fermi energy E_F . In contrast, there are very few studies on “narrow” resonances[1], for there is a general belief that at all temperatures, the effect of a resonance could only be felt within its width ΔB , and to stabilize the magnetic field within the width of a narrow resonance is a taunting task.

Here, we show that such general view is incorrect. We shall see that in both high and low temperature regimes, the interaction energy associated with a narrow resonance is comparable to that of the unitary gas even when the system is many widths away from the resonance, as long as the distance from resonance $\gamma(B - B_\infty)$ is within the relevant energy scale of the system \mathcal{E}^* , i.e. $\Delta B \ll \gamma(B - B_\infty) \ll \mathcal{E}^*$, where \mathcal{E}^* is the Fermi energy E_F at low temperatures, and temperature T at high temperatures. More specifically, we shall show that at low temperatures, the interaction energy is a substantial fraction of the total energy; and that at high temperatures, the second virial coefficient b_2 at resonance is twice as large as that of a wide resonance, and remains sizable beyond its width. There is, however, a major difference between these two resonances. For wide resonances, the interaction energy of the scattering states ($\epsilon^{extended}$) is antisymmetric about the resonance, whereas it is highly asymmetric for narrow resonances – strongly attractive on the atomic side, and weakly repulsive on the molecular side. Such differences are caused by a resonance structure of the phase shift unique to narrow resonances. The aforementioned large interaction energy and strong asymmetry should be observable experimentally.

Wide and narrow resonance: Currently, there are two

schemes to classify wide and narrow resonances. The first (denoted as **(A)**) is based on the property of the two-body systems[2]. The second (denoted as **(B)**) makes use of the Fermi energy E_F of the many-body system [3–6]. They are not equivalent. To discuss their differences and relations, recall that the s-wave scattering length as a function of magnetic field is $a_s(B) = a_{bg} \left(\frac{B - B_0}{B - B_\infty} \right)$, or

$$a_s(B) = a_{bg} \left(1 - \frac{\gamma \Delta B}{\gamma(B - B_\infty)} \right) \equiv a_{bg} - \frac{r_* E_*}{\gamma(B - B_\infty)} \quad (1)$$

where a_{bg} is the background scattering length, B_0 and B_∞ are the magnetic fields at which a_s vanishes and diverges respectively; $\Delta B \equiv B_0 - B_\infty$ is the width of the resonance, γ is the magnetic moment difference between the atom and closed molecular state. It can be shown from two-channel models that $a_{bg}\gamma\Delta B > 0$ [7]. We can then define a length r_* as $\hbar^2/(mr_*) \equiv a_{bg}\gamma\Delta B$, and energy scale $E_* = \hbar^2/mr_*^2$. The last equality in Eq.(1) is a more general representation as it applies to the case $a_{bg} = 0$. If r_* is non-zero, then ΔB must diverge as a_{bg} vanishes. The region $B > B_\infty$ and $B < B_\infty$ will be referred to as the “atomic” and “molecular” side of the resonance, as a bound state appears when $B < B_\infty$.

Eq.(1) shows that the resonance is characterized by the energies $E_{bg} = \hbar^2/(ma_{bg}^2)$ and $\gamma\Delta B$. Their ratio will be denoted as

$$\alpha \equiv \frac{\gamma\Delta B}{E_{bg}} = \frac{a_{bg}}{r_*}. \quad (2)$$

In scheme **(A)**[2], wide and narrow resonance (denoted as W_A and N_A) correspond to

$$W_A : |\alpha| \gg 1, \quad N_A : |\alpha| \ll 1, \quad \text{Scheme (A)}. \quad (3)$$

In scheme **(B)**[3–6], they correspond to

$$W_B : r_* k_F \ll 1, \quad N_B : r_* k_F \gg 1, \quad \text{Scheme (B)}. \quad (4)$$

Note that **(B)** makes no reference to the width ΔB , nor to the background scattering length a_{bg} . In fact, many theoretical studies of “narrow” resonances make use a “simple” two-channel models with $a_{bg} = 0$, which automatically sends $|\Delta B|$ to ∞ [8]. By construction, such

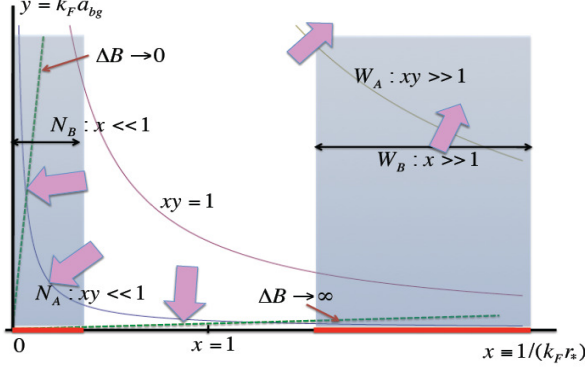


FIG. 1. Wide and narrow resonances in the space $((k_F r_*)^{-1}, k_F a_{bg})$: W_A and N_A are the regions above and below the curve $xy = 1$ respectively. W_B and N_B are the regions where $x \gg 1$ and $x \ll 1$. The dotted lines are contours of constant ΔB . The thick red lines on the x -axis correspond to the simple two channel models[8], (see text). This phase diagram is for $a_{bg} > 0$. The full phase diagram including $a_{bg} < 0$ is mirror symmetric about the x -axis. For ^6Li at density $n_\uparrow = n_\downarrow = 5 \times 10^{14} \text{cm}^{-3}$, its wide resonance at 834.1G has $a_{bg} = -1405 a_B$, $\Delta B = -300 \text{G}$, $|a_{bg}|/r_* = 2.8 \times 10^3$, $r_* k_F = 8 \times 10^{-4}$, and $k_F |a_{bg}| = 2.3$. Its narrow resonance at 543.25G has $a_{bg} = 61.6 a_B$, $\Delta B = 0.1 \text{G}$, $a_{bg}/r_* = 0.002$, $r_* k_F = 50$, and $k_F a_{bg} = 0.1$ [2].

models can not produce the basic features of $a_s(B)$ for all resonances that are experimentally regarded as narrow, which all have very narrow width ΔB . On the other hand, as we shall see, such simple model does capture the essential feature of the phase shift, which is what is needed to account for the energy of the extended states, see footnote [9] later. Yet when it comes to many-body features that depend crucially on a_{bg} and ΔB , it is necessary to go beyond these simple models. To make contact with experiments, we shall consider $a_{bg} \neq 0$, with $|a_{bg}| k_F < 1$, i.e. the background scattering is not near resonance.

The relation between scheme (A) and (B) in the space of (r_*^{-1}, a_{bg}) is shown in Figure 1. Defining the dimensionless quantities $x = (k_F r_*)^{-1}$ and $y = k_F a_{bg}$, scheme (A) and (B) correspond to the follow regions in the (x, y) plane, (W_A : $xy = \alpha \gg 1$; N_A : $xy \ll 1$) and (W_B : $x \gg 1$; N_B : $x \ll 1$). (See Figure 1). The contours of constant ΔB are denoted by dotted straight lines, whose slope y/x is the ratio $2E_F/(\gamma \Delta B)$. The regions where the simple two channel model (with $a_{bg} = 0$ and hence $\Delta B = \infty$) applies are along the x -axis, indicated by thick red lines.

Origin of Strong Interaction: The effects of interaction can be found in the two body phase shift $\delta(k)$. From the general two channel model which includes the background scattering length a_{bg} , one can calculate the scattering amplitude $f(k) = 1/(k \cot \delta(k) - ik)$ and finds that the energy dependent scattering length $a(k)$, defined as

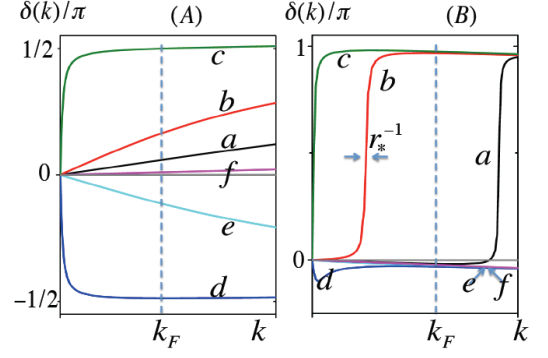


FIG. 2. $\delta(k)$ vs k for wide and narrow resonances: The labels (a,b,c,d,e,f) correspond to (a : $B \gg B_\infty$; b : $B > B_\infty$; c : $B = B_\infty + 0^+$; d : $B = B_\infty - 0^+$; e : $B < B_\infty$; f : $B \ll B_\infty$). For wide resonance with $a_{bg} < 0$, (Fig. 2A), we have $\delta(k) = -\arctan(ka_s(B))$. Near resonance, (c and d), $\delta(k)$ approaches a step function of height $\pm\pi/2$. Far from resonance, (a and f) $\delta(k)$ reduces to $\delta(k) = -\arctan(ka_{bg}) \sim -ka_{bg}$, and $|ka_{bg}| \ll 1$. For narrow resonance with $a_{bg} > 0$, (Fig. 2B), when $B > B_\infty$, $\delta(k)$ approaches a step function of height π with a width $1/r_*$. For $B < B_\infty$, $\delta(k)$ quickly reduces to $-ka_{bg}$ as $B_\infty - B$ exceeds ΔB , and $|ka_{bg}| \ll 1$. The $\delta(k)$ for both e and f are essentially identical.

$k \cot \delta(k) \equiv -1/a(k)$, has the general form[7]

$$a(k) = a_{bg} \left(\frac{\hbar^2 k^2/m - \gamma(B - B_0)}{\hbar^2 k^2/m - \gamma(B - B_\infty)} \right), \quad (5)$$

and $a(k = 0)$ is the s-wave scattering length whose magnetic field dependence is given in Eq.(1). It is straightforward to show from Eq.(5) that

$$\tan \delta(k) = -ka_{bg} - \frac{k \hbar^2/m r_*}{\hbar^2 k^2/m - \gamma(B - B_\infty)}. \quad (6)$$

Since $\delta(k)$ is defined up to modulo π , its magnetic field dependence is to be determined by Levinson's theorem – that it is a continuous function of B until a bound state emerges, in which case it jumps down by $-\pi$. Applying the conditions in Eq.(3) and (4) to Eq.(6), we have plotted $\delta(k)$ for wide and narrow resonance for various magnetic fields in Figure 2A and 2B respectively.

Figure 2A shows that as one approaches a wide resonance from the atomic side, $\delta(k)$ starts from the form $-a_{bg}k$ (which is much less than 1 for k up to a few k_F); and turns into a step function of height $\pi/2$. In the latter case, all energy states to gain a $\pi/2$ phase shift, leading to a large negative interaction energy. Crossing the resonance to the molecular side, $\delta(k)$ jumps down by $-\pi$ due to the appearance of a bound state, leading to a repulsive energy on the molecular side.

For a narrow resonance, $\delta(k)$ is essentially a step function of height π located at energy $\hbar^2 k^2/m = \gamma(B - B_\infty)$ with a width $\Delta k = 1/r_*$, as shown in Figure 2B. We

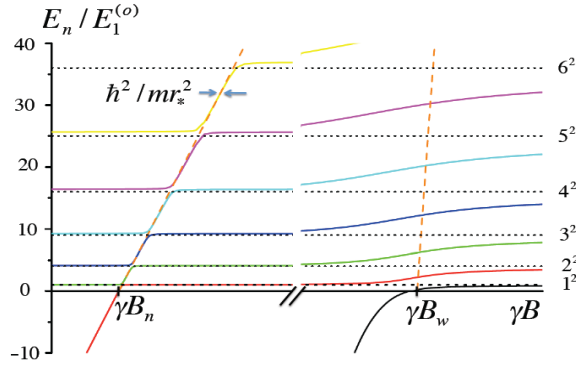


FIG. 3. The evolution of energy level as B passes through a wide resonance (at B_w) and a narrow resonance (at B_n): For wide resonance, the transition from $k_n^{(o)}$ to $k_{n-1}^{(o)}$ is completed only half way at resonance. For narrow resonance, due to the narrow width, this transition is completed on the atomic side of the resonance, before the bound state is formed. Passing the resonance to the molecular side, ($B < B_\infty$), $\delta(k) \ll 1$ and the scattering state has little interaction energy.

can see that even when $\gamma\Delta B \ll \gamma(B - B_\infty) \ll 2E_F = \hbar^2 k_F^2/m$ or T , i.e. the system is far beyond the width of the resonance, all states with energy above $\gamma(B - B_\infty)$ are phase shifted by π , (twice the value of wide resonance); thereby generating considerable interaction energy. As B passes through to the molecular side, $\delta(k)$ shifts π down to zero due to the the presence of a bound state, leading to a small interaction energy for the scattering state.

A more detailed understanding of the origin of interaction energy can be obtained from the evolution of the energy levels as they pass through a resonance. Fig.3 shows such an evolution for the passage of a wide followed by that of a narrow resonance, as we note that many atoms have narrow resonances close to wide ones. The energy levels $\hbar^2 k^2/m$ are calculated through standard methods by setting the wavefunction of the extended state $\psi(r) = r^{-1}\sin(kr + \delta(k))$ to zero at a large distance $r = R$, which gives $k_n = k_n^{(o)} - \delta(k_n)/R$, where $k_n^{(o)} = n\pi/R$, $n = 1, 2, \dots$ are the momenta of non-interacting systems.

As one moves from right to left in Fig.3, the energy levels shift down from $k_n^{(o)}$ to $k_{n-1}^{(o)}$, with the lowest level developed into a bound state. For the wide resonance, the transition from $k_n^{(o)}$ to $k_{n-1}^{(o)}$ ($n \geq 1$) is only completed half way at resonance, due to the fact that $\delta = \pi/2$ at resonance. In contrast, for a narrow resonance located at B_n , $k_\ell^{(o)}$ shifts down to $k_{\ell-1}^{(o)}$ rapidly (due to the narrow width) on the atomic side of the resonance ($B > B_n$) at $\gamma(B - B_n) = \hbar^2 k_\ell^{(o)2}/m$. As a result, all momentum states with energies above $\gamma(B - B_n)$ are phase shifted by π , thereby generating considerable interaction energy. On the other hand, on the molecular side of the resonance ($B < B_n$), the phase shift becomes very small (see curves

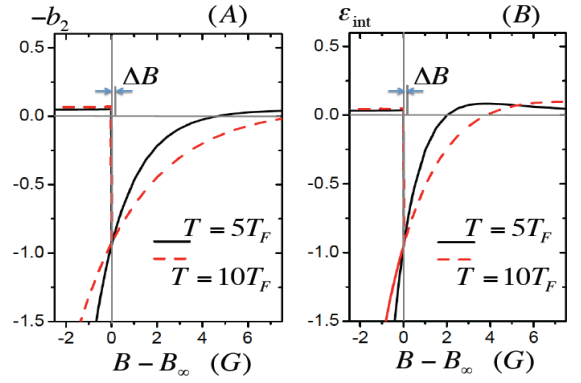


FIG. 4. $-b_2$ (A) and ϵ_{int} (B) as a function of magnetic field: We consider a gas of ^6Li at 543.25G with parameters given the caption of Figure 1. In this case, $T_F = 40\mu\text{K} \sim 3\gamma\Delta B$ (in temperature units). Note that b_2 remains sizable even when $B - B_\infty \sim 2\text{G} \sim 20\Delta B$. This is because we are at $T = 5T_F \sim 15\gamma\Delta B$ or larger, where thermal sampling extends over an energy range of several T . When the system is in equilibrium, it follows the lower curve (or the “lower branch”). At resonance and in the limit $\Delta B \rightarrow 0$, $b_2 \rightarrow 1$. On the molecular side, $B < B_\infty$, $\epsilon_{int}^{extended}$ (for the upper branch) is given by the flat line, given by the small value $\delta \sim -kabg$.

d, e, f in figure 2B). The energy levels therefore becomes very close to the non-interacting energy level as shown in Fig.3, implying little interaction energy on the molecular side.

Recently, it has been shown in ref.[10] that the energy dependence of the phase shift of “narrow” resonances can lead to a high T_c/T_F ratio. The resonance they studied, however, has $r_*k_F < 1$. It is therefore not narrow in the present as well as previous classifications[3–6]. The origin of strong interaction, and the implication of the π -jump in the phase shift of narrow resonances were not discussed in ref.[10].

Virial expansion: A useful gauge of the interaction effects of narrow resonances is to compare them with those of wide resonances. At high temperatures, the interaction energy of a gas with density n is given by

$$\epsilon_{int}(T, n) = \frac{3Tn}{2} \left(\frac{n\lambda^3}{\sqrt{2}} \right) \left[-b_2 + \frac{2T}{3} \frac{\partial b_2}{\partial T} \right], \quad (7)$$

where b_2 is the second virial coefficient. It consists of two contributions, $b_2 = b_2^{bound} + b_2^{extended}$, where $b_2^{bound} = \sum_\alpha e^{E_\alpha^b/T}$ is the partition function of the bound states $\{E_\alpha^b\}$ (labelled by α) and

$$b_2^{extended} = \int_0^\infty \frac{dk}{\pi} \frac{d\delta(k)}{dk} e^{-\hbar^2 k^2/mT} \quad (8)$$

is the change of the partition function of the scattering states. Their corresponding contributions to interaction energy will be denoted as $\epsilon_{int} = \epsilon_{int}^{bound} + \epsilon_{int}^{extended}$ respectively. For wide resonances, b_2 and ϵ_{int} have been

studied in ref.[11]. In Figure 4A and 4B, we have plotted both $-b_2$ and ϵ_{int} for narrow resonance. They are the curves that bend downward. The flat lines that exist only on the molecular side of the resonance are $b_2^{extended}$ and $\epsilon_{int}^{extended}$. They contain only the contributions from the scattering state and are sometimes referred to as “upper” or “repulsive” branch quantities. The full b_2 and ϵ_{int} consisting of both scattering state and bound state contributions are referred to as “lower” or “attractive” branch quantities. Note that the interaction energy of the repulsive branch is close to 0, $\epsilon_{int}^{extended} \sim 0$, whereas that of the attractive branch near resonance is large and negative. This is due to the special property of $\delta(k)$ discussed in the previous section. Comparing the interaction energy of narrow resonance (in Fig.4) with that of wide resonance[11], we note that the range of B field over which b_2 is non-zero is much smaller than that of the wide resonance. Despite that, the large value of b_2 still extends far beyond the width of the resonance, and its value at resonance is 1, twice of that of a wide resonance. This is a consequence of the π jump of $\delta(k)$ at energy $\hbar^2 k^2/m = \gamma(B - B_\infty)$ shown in Fig.2B.

Interaction energy at low temperatures: To extend the study of upper and lower branch energy to lower temperatures, we apply a generalization of Nozieres-Schmitt-Rink (NSR) method to calculate the energy of the system[12]. Central to the NSR method is the T -matrix ($T(\mathbf{q}, \omega)$) in a medium for a pair of fermions with total momentum \mathbf{q} . In the case of narrow resonance, it is

$$\frac{1}{T(\mathbf{q}, \omega)} = \frac{m}{4\pi a_s(k)} - \frac{1}{\Omega} \sum_{\mathbf{p}} \left(\frac{\gamma(\mathbf{p}, \mathbf{q})}{\omega_+ - \omega(q) - \frac{p^2}{m}} - \frac{1}{\frac{p^2}{m}} \right), \quad (9)$$

where $k^2/m = \omega - \omega(q)$, $\omega(q) = q^2/(4m) - 2\mu$; $\omega_+ = \omega + i0^+$, $\gamma(\mathbf{p}, \mathbf{q}) = 1 - n_F(\xi_{\mathbf{q}/2+\mathbf{p}}) - n_F(\xi_{\mathbf{q}/2-\mathbf{p}})$ describes the Pauli blocking effect of the medium, $\xi_{\mathbf{p}} = p^2/(2m) - \mu$ and we have set $\hbar = 1$. From the phase of inverse T -matrix, $\delta(\mathbf{q}, \omega) = \arg T^{-1}(\mathbf{q}, \omega)$, one can calculate the pressure, the energy density, and equation of state of the attractive branch and the repulsive branch. In Fig.5, we have plotted the result for the interaction energy for both the attractive and repulsive branch across resonance at a temperature $T = 0.5T_F$. The behaviors of both branches are similar to those found from the virial expansion (Fig.3). The energy scale, however, is very different. One sees that the interaction energy is as much as 50% of the total energy of an ideal Fermi gas right at resonance, and can be as high as 30 – 40% *even beyond the width of the resonance*. In Fig. 5, one also see that the interaction energy is only significant when the distance from resonance $\gamma(B - B_\infty)$ is within $2E_F$, as discussed previously.

After the posting of an earlier version of this paper which does not include the low temperature results, Ken O’hara’s group has performed rf spectroscopy studies on the narrow resonance of ^6Li at 543.25G and has found the asymmetry of interaction energy ϵ_{int} on different side of

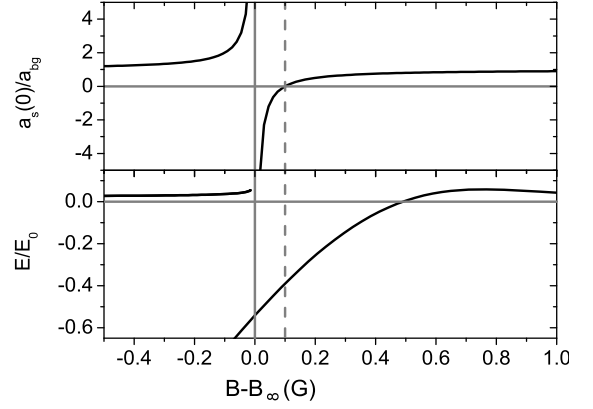


FIG. 5. S-wave scattering length(upper panel) and interaction energy (lower panel) of ^6Li across 543.25G narrow resonance at low temperatures. We have $T = 0.5T_F$, $T_F = 3\gamma\Delta B$ as in Fig.4. Other parameters are the same as those given in the caption of Fig.1. E_0 is the energy for a non-interacting Fermi gas at the same temperature. The dashed line indicates the width ΔB of the resonance. The downward turning curve and the flat curve are the energies of the lower and upper branch respectively. The interaction energy of the lower branch reaches 50% of the free fermion energy at resonance, and remains sizable beyond the width of the resonance.

the resonance[13]. We note that ϵ_{int} can also be determined exactly (free of the modeling by specific theories) from in situ density measurements[14].

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